

# Structure and asymptotics for Motzkin numbers modulo primes using automata

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## Abstract

We establish a lower bound of  $\frac{2}{p(p-1)}$  for the asymptotic density of the Motzkin numbers divisible by a general prime number  $p \geq 5$ . We provide a criteria for when this asymptotic density is actually 1. We also provide a partial characterisation of those Motzkin numbers which are divisible by a prime  $p \geq 5$ . All results are obtained using the automata method of Rowland and Yassawi.

## 1 Introduction

The Motzkin numbers  $M_n$  are defined by

$$M_n := \sum_{k \geq 0} \binom{n}{2k} C_k$$

where  $C_k$  are the Catalan numbers.

There has been some work in recent years on analysing the Motzkin numbers  $M_n$  modulo primes and prime powers. Deutsch and Sagan [3] provided a characterisation of Motzkin numbers divisible by 2, 4 and 5. They also provided a complete characterisation of the Motzkin numbers modulo 3 and showed that no Motzkin number is divisible by 8. Eu, Liu and Yeh [4] reproved some of these results and extended them to include criteria for when  $M_n$  is congruent to  $\{2, 4, 6\} \pmod{8}$ . Krattenthaler and Müller [6] established identities for the Motzkin numbers modulo higher powers of 3 which include the modulo 3 result of [3] as a special case. Krattenthaler and Müller [5] have more recently extended this work to a full characterisation of  $M_n \pmod{8}$  in terms of the binary expansion of  $n$ . The results in [6] and [5] are obtained by expressing the generating function of  $M_n$  as a polynomial involving a special function. Rowland and Yassawi [7] investigated  $M_n$  in the general setting of automatic sequences. The values of  $M_n$  (as well as other sequences) modulo prime powers can be computed via automata. Rowland and Yassawi provided algorithms for creating

the relevant automata. They established results for  $M_n$  modulo small prime powers, including a full characterisation of  $M_n$  modulo 8 (modulo  $5^2$  and  $13^2$  are available from Rowland's website). They also established that 0 is a forbidden residue for  $M_n$  modulo 8,  $5^2$  and  $13^2$ . In theory the automata can be constructed for any prime power but computing power and memory quickly becomes a barrier. For example, the automata for  $M_n$  modulo  $13^2$  has over 2000 states. Rowland and Yassawi also went on to describe a method for obtaining asymptotic densities of  $M_n$ . We have previously [2] used Rowland and Yassawi's work to analyse Motzkin numbers modulo specific primes up to 29. This current paper will deal with a general prime  $p$ . It turns out that the behaviour of  $M_n$  modulo a general prime is similar to the behaviour modulo small primes.

We will use Rowland and Yassawi's automata to establish a lower bound on the asymptotic density of the set of  $M_n$  divisible by a general prime  $p \geq 5$ . This lower bound is  $\frac{2}{p(p-1)}$ . As shown in [2] the asymptotic density is actually 1 for some primes, e.g.  $p = 7, 17, 19$ . We will also make note of some structure results that appear from an examination of the relevant state diagrams of the automata. In particular  $M_n \equiv 0 \pmod p$  when  $n$  takes certain forms depending on the prime  $p$ . This generalises results that had already been shown to hold for particular small primes as mentioned in the previous paragraph. It is found that the behaviour of  $M_n \pmod p$  depends to some extent on the value of  $p \pmod 6$  - this is either +1 or -1 of course.

Table 1 summarises the results that will be presented in subsequent sections. Firstly, we will explain some of the definitions we have used in this paper.

The asymptotic density of a subset  $S$  of  $\mathbb{N}$  is defined to be

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n \in S : n \leq N\}$$

if the limit exists, where  $\#S$  is the number of elements in a set  $S$ . For a prime number  $p \geq 5$  we mainly will be studying the asymptotic density of the set

$$S_p(0) = \{n \in \mathbb{N} : M_n \equiv 0 \pmod p\} . \quad (1)$$

However, for any  $x \in \mathbb{N}$  we define the set

$$S_p(x) = \{n \in \mathbb{N} : M_n \equiv x \pmod p\} .$$

For a number  $p$ , we write the base  $p$  expansion of a number  $n$  as

$$[n]_p = \langle n_r n_{r-1} \dots n_1 n_0 \rangle$$

where  $n_i \in [0, p-1]$  and

$$n = n_r p^r + n_{r-1} p^{r-1} + \dots + n_1 p + n_0 .$$

Prime	Density	Values of $n$ such that $M_n \equiv 0 \pmod{p}$
$p \equiv 1 \pmod{6}$	$\geq \frac{2}{p(p-1)}$	$n = (pi + 1)p^k - 2$ for $i \geq 0$ and $k \geq 1$ . $n = (pi + p - 1)p^k - 1$ for $i \geq 0$ and $k \geq 1$ .
$p \equiv -1 \pmod{6}$	$\geq \frac{2}{p(p-1)}$	$n = (pi + 1)p^{2k} - 2$ for $i \geq 0$ and $k \geq 1$ . $n = (pi + p - 2)p^{2k+1} - 2$ for $i \geq 0$ and $k \geq 0$ . $n = (pi + 2)p^{2k+1} - 1$ for $i \geq 0$ and $k \geq 0$ . $n = (pi + p - 1)p^{2k} - 1$ for $i \geq 0$ and $k \geq 1$ .

Table 1: Table of results

Binomial coefficients are prominent in this paper. Here the binomial coefficient  $\binom{n}{m}$  is defined to be 0 when  $m > n$  or when either  $n$  or  $m$  is negative.

## 2 Background on Motzkin numbers modulo primes

As mentioned in the introduction there have been results which characterise  $M_n$  modulo primes  $p \leq 29$ . We collect these below to allow comparison with the results for a general prime.

**Theorem 1.** (Theorem 5.5 of [4]). *The  $n$ th Motzkin number  $M_n$  is even if and only if*

$$n = (4i + \epsilon)4^{j+1} - \delta \text{ for } i, j \in \mathbb{N}, \epsilon \in \{1, 3\} \text{ and } \delta \in \{1, 2\}.$$

**Theorem 2.** (Corollary 4.10 of [3]). *Let  $T(01)$  be the set of numbers which have a base-3 representation consisting of the digits 0 and 1 only. Then the Motzkin numbers satisfy*

$$M_n \equiv \begin{cases} -1 \pmod{3} & \text{if } n \in 3T(01) - 1, \\ 1 \pmod{3} & \text{if } n \in 3T(01) \text{ or } n \in 3T(01) - 2, \\ 0 \pmod{3} & \text{otherwise.} \end{cases}$$

**Theorem 3.** (Theorem 5.4 of [3]). *The Motzkin number  $M_n$  is divisible by 5 if and only if  $n$  is one of the following forms*

$$(5i + 1)5^{2j} - 2, (5i + 2)5^{2j-1} - 1, (5i + 3)5^{2j-1} - 2, (5i + 4)5^{2j} - 1$$

where  $i, j \in \mathbb{N}$  and  $j \geq 1$ .

The above results and others have been used to establish asymptotic densities of the sets  $S_q(0)$  for  $q = 2, 4, 8$  and also for primes up to 29 - see [7], [1] and [2]. In particular the asymptotic density of  $S_2(0)$  is  $\frac{1}{3}$  ([7] example 3.12), the asymptotic density of  $S_4(0)$  is  $\frac{1}{6}$  ([7] example 3.14), the asymptotic density of  $S_q(0)$  is 1 for  $q \in \{3, 7, 17, 19\}$  [1] and [2], the asymptotic density of  $S_q(0)$  is  $\frac{2}{q(q-1)}$  for  $q \in \{5, 11, 13, 23\}$  [1] and [2]. The asymptotic density of  $S_{29}(0)$  satisfies  $\frac{2}{29 \cdot 28} < S_{29}(0) < 1$ .

There are 2 questions which we will investigate in this article. Firstly, what is the asymptotic density of  $S_p(0)$  for general primes  $p \geq 5$ ? Secondly, what structural features are evident in the distribution of  $M_n \bmod p$ ? The investigation will proceed by constructing an automaton for a general prime following the instructions from [7] (Algorithm 1). The state diagram for the automaton provides an excellent tool for analysing the behaviour of  $M_n \bmod p$ .

### 3 Three useful series and some modular identities involving binomial coefficients

There are 3 series that will appear regularly during the construction of the automaton. We will therefore devote this section to a discussion of these series which are interesting in their own right. We define the series as follows:

$$a_n := \sum_{k \geq 0} (-1)^k \binom{n-k}{k} \quad (2)$$

$$b_n := \sum_{k \geq 0} (-1)^k \binom{n-k}{k} k \quad (3)$$

$$c_n := \sum_{k \geq 0} (-1)^k \binom{n-k}{k} k^2. \quad (4)$$

**Theorem 4.**

$$a_n = \cos\left(\frac{\pi n}{3}\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\pi n}{3}\right)$$

$$b_n = \frac{2n}{3} \cos\left(\frac{\pi n}{3}\right) - \frac{2}{3\sqrt{3}} \sin\left(\frac{\pi n}{3}\right)$$

and

$$c_n = \frac{n}{3} (n-1) \cos\left(\frac{\pi n}{3}\right) - \frac{1}{3\sqrt{3}} (n^2 + n - 2) \sin\left(\frac{\pi n}{3}\right)$$

*Proof.* The three series each satisfy a linear difference equation. The solutions to these equations can be derived using standard methods. For  $a_n$  we have  $a_0 = 1$ ,  $a_1 = 1$  and

$$\begin{aligned} a_{n+1} &= \sum_{k \geq 0} (-1)^k \binom{n+1-k}{k} \\ &= \sum_{k \geq 0} (-1)^k \binom{n-k}{k} + \sum_{k \geq 0} (-1)^k \binom{n-k}{k-1} \\ &= \sum_{k \geq 0} (-1)^k \binom{n-k}{k} - \sum_{k \geq 0} (-1)^k \binom{n-1-k}{k} \end{aligned}$$

where we have used the binomial identity

$$\binom{s+1}{r} = \binom{s}{r} + \binom{s}{r-1}. \quad (5)$$

So  $a_n$  satisfies the difference equation

$$a_{n+1} - a_n + a_{n-1} = 0 \quad (6)$$

The solution of this difference equation is given in the statement of the theorem. The initial values of  $b_n$  are  $b_0 = 0$ ,  $b_1 = 0$  and, using the identity (5) again, we can show  $b_n$  satisfies the non-homogeneous difference equation

$$b_{n+1} - b_n + b_{n-1} = -a_{n-1}. \quad (7)$$

Finally,  $c_0 = c_1 = 0$  and  $c_n$  satisfies the non-homogeneous difference equation

$$c_{n+1} - c_n + c_{n-1} = -a_{n-1} - 2b_{n-1}. \quad (8)$$

□

The results can be extended to the series

$$\sum_{k \geq 0} (-1)^k \binom{n-k}{k} k^m.$$

where  $m$  is arbitrary but we only need the equations for  $a_n$ ,  $b_n$  and  $c_n$ . We list here some of the properties of the series which will be needed later. Firstly,  $a_n$  is a periodic sequence with period 6. Starting with  $n = 0$  the sequence  $a_n$  is  $1, 1, 0, -1, -1, 0, 1, 1, \dots$ . For any  $n \in \mathbb{N}$ ,

$$a_{n-1} = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{6}; \\ -1 & \text{if } n \equiv -1 \pmod{6}; \end{cases}$$

$$a_n = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{6}; \\ 0 & \text{if } n \equiv -1 \pmod{6}; \end{cases}$$

$$b_n = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod{6}; \\ \frac{n+1}{3} & \text{if } n \equiv -1 \pmod{6}; \end{cases}$$

and

$$c_n = \begin{cases} -\frac{(n-1)}{3} & \text{if } n \equiv 1 \pmod{6}; \\ \frac{n^2-1}{3} & \text{if } n \equiv -1 \pmod{6}. \end{cases}$$

The above identities for  $a_n$  show that if  $p$  is prime (so  $p \equiv \pm 1 \pmod{6}$ )

$$a_{p-1} - 2a_p + 1 = 0 \tag{9}$$

and

$$b_p + c_p = \begin{cases} 0 & \text{if } p \equiv 1 \pmod{6}; \\ \frac{p(p+1)}{3} & \text{if } p \equiv -1 \pmod{6}. \end{cases} \tag{10}$$

We will also need the following identities

$$a_p - b_p - 1 + \frac{1}{2}b_{p+1} + \frac{1}{2}c_{p+1} - b_{p+2} - c_{p+2} = \begin{cases} \frac{p(p+5)}{6} & \text{if } p \equiv 1 \pmod{6}; \\ \frac{p(p+1)}{6} - 1 & \text{if } p \equiv -1 \pmod{6}. \end{cases} \tag{11}$$

$$-a_{p-1} + a_p + b_p - 2b_{p+1} - 2a_{p+1} + 1 = \begin{cases} p+2 & \text{if } p \equiv 1 \pmod{6}; \\ -p-1 & \text{if } p \equiv -1 \pmod{6}. \end{cases} \tag{12}$$

There are also a few modular identities that will be useful in simplifying some of the equations that will appear later. Firstly, for  $k, l \in \mathbb{N}$ ,

$$\binom{p-1-k}{l} \equiv (-1)^l \binom{k+l}{k} \pmod{p}. \tag{13}$$

In particular,

$$\binom{p-1}{l} \equiv (-1)^l \pmod{p}$$

$$\begin{aligned}\binom{p-2}{l} &\equiv (-1)^l(l+1) \pmod{p} \\ \binom{p-3}{l} &\equiv (-1)^l \binom{l+2}{2} \equiv \frac{1}{2}(-1)^l(l+2)(l+1) \pmod{p}.\end{aligned}$$

## 4 Background on automata for $M_n \pmod{p}$

Rowland and Yassawi showed in [7] that the behaviour of sequences such as  $M_n \pmod{p}$  can be studied by the use of finite state automata. The automaton has a finite number of states and rules for transitioning from one state to another. In the form described in [7] each state  $s$  is represented by a polynomial in 2 variables  $x$  and  $y$ . Each state has a value obtained by evaluating the polynomial at  $x = 0$  and  $y = 0$ . All calculations are made modulo  $p$ . For the Motzkin case the initial state  $s_1$  is represented by the polynomial

$$R(x, y) = y(1 - xy - 2x^2y^2 - 2x^2y^3). \quad (14)$$

New states are constructed by applying the Cartier operator  $\Lambda_{d,d}$  to the polynomials

$$s_i * Q^{p-1}(x, y)$$

for  $d \in \{0, 1, \dots, p-1\}$  where  $\{s_i\}$  are the already calculated states and the polynomial  $Q$  is defined by

$$Q(x, y) = x^2y^3 + 2x^2y^2 + x^2y + xy + x - 1 = x^2y(y+1)^2 + x(y+1) - 1. \quad (15)$$

The Cartier operator is a linear map on polynomials defined by

$$\Lambda_{d_1, d_2} \left( \sum_{m, n \geq 0} a_{m, n} x^m y^n \right) = \sum_{m, n \geq 0} a_{pm+d_1, pn+d_2} x^m y^n. \quad (16)$$

Since the Cartier operator maintains or reduces the degree of the polynomial and there are only finitely many polynomials modulo  $p$  of each degree, all states of the automaton are obtained within a known finite time. It will be seen later that the automaton has at most  $p + 6$  states. If

$$\Lambda_{d,d}(s * Q^{p-1}) = t$$

for states (i.e. polynomials)  $s$  and  $t$  then the transition from state  $s$  to state  $t$  under the input  $d$  is part of the automaton.

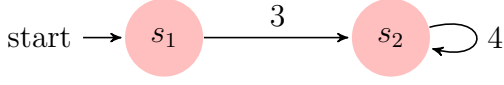


Figure 1: Example of transition from state  $s_1$  to state  $s_2$  and a loop.

To calculate  $M_n \bmod p$ ,  $n$  is first represented in base  $p$ . The base  $p$  digits of  $n$  are fed into the automaton starting with the least significant digit. The automaton starts at the initial state  $s_1$  and transitions to a new state as each digit is fed into it. The value of the final state after all  $n$ 's digits have been used is equal to  $M_n \bmod p$ . Refer to [7] for more details.

In the remainder of this article we will provide details of the automata for a general prime  $p \geq 5$ . We will provide the polynomials and values for the states and the relevant transitions between states. States are listed as  $s_1, s_2, \dots$ . Transitions, when provided, will be in the form  $(s, j) \rightarrow t$  which means that if the automaton is in state  $s$  and receives digit  $j$  then it will move to state  $t$ . We will call a state  $s$  a **loop** state if all transitions from  $s$  go to  $s$  itself, i.e.  $(s, j) \rightarrow s$  for all choices of  $j$ .

States and transitions are represented visually in the form of a directed graph. For example, figure 1 represents an automaton which moves from state  $s_1$  to state  $s_2$  when it receives the digit 3. It also moves from state  $s_2$  to state  $s_2$  (i.e. loops) if it is in state  $s_2$  and receives a digit 4.

## 5 Preliminary calculations

Before we start constructing the automata it will be convenient to first precompute  $\Lambda_{d,d}(s(x, y) * Q(x, y)^{p-1})$  for some simple choices of the polynomial  $s$ . The relevant results are contained in tables 2 and 3. When reading the table note that  $\binom{n}{m} = 0$  for  $m < 0$ . We will go through a few of the calculations from tables 2 and 3.

Firstly, the polynomial  $Q^{p-1}$  can be written as

$$\begin{aligned}
 Q^{p-1}(x, y) &= \left( x^2 y (y+1)^2 + x(y+1) - 1 \right)^{p-1} \\
 &= \sum_{k \geq 0} \binom{p-1}{k} x^{2k} y^k (y+1)^{2k} \left( x(y+1) - 1 \right)^{p-1-k} \\
 &= \sum_{k, l \geq 0} \binom{p-1}{k} x^{2k} y^k (y+1)^{2k} \binom{p-1-k}{l} x^l (y+1)^l (-1)^{p-1-k-l}
 \end{aligned}$$



State $s$	$\Lambda_{d,d}(s * Q^{p-1})$
1	$1$ for $\mathbf{0} \leq \mathbf{d} \leq \mathbf{1}$ $\sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k} (-1)^d$ for $\mathbf{2} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{3}$ $b_p$ for $\mathbf{d} = \mathbf{p} - \mathbf{2}$ $a_{p-1}$ for $\mathbf{d} = \mathbf{p} - \mathbf{1}$
$y$	$0$ for $\mathbf{d} = \mathbf{0}$ $1$ for $\mathbf{d} = \mathbf{1}$ $\sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k+1} (-1)^d$ for $\mathbf{2} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{3}$ $-\sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k+1} + xy + xy^2$ $= -a_p - b_p + 1 + xy + xy^2$ for $\mathbf{d} = \mathbf{p} - \mathbf{2}$ $\sum_{k \geq 0} \binom{p-1-k}{k} \binom{p-1}{k+1} = -a_{p-1}$ for $\mathbf{d} = \mathbf{p} - \mathbf{1}$
$x^2 y^2$	$-\sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k} xy = b_p xy$ for $\mathbf{d} = \mathbf{0}$ $\sum_{k \geq 0} \binom{p-1-k}{k} \binom{p-1}{k} xy = a_{p-1} xy$ for $\mathbf{d} = \mathbf{1}$ $\sum_{k \geq 0} \binom{p-1-k}{d-2-2k} \binom{d-2}{k} (-1)^d$ for $\mathbf{2} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{1}$
$xy^2$	$\sum_{k \geq 0} \binom{p-1-k}{k} \binom{p-1}{k+1} xy = -a_{p-1} xy$ for $\mathbf{d} = \mathbf{0}$ $0$ for $\mathbf{d} = \mathbf{1}$ $\sum_{k \geq 0} \binom{p-1-k}{d-1-2k} \binom{d-1}{k+1} (-1)^{d+1}$ for $\mathbf{2} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{3}$ $-\frac{1}{2}(c_{p+1} + b_{p+1})$ for $\mathbf{d} = \mathbf{p} - \mathbf{2}$ $-\sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k+1} + xy + xy^2$ $= -a_p - b_p + 1 + xy + xy^2$ for $\mathbf{d} = \mathbf{p} - \mathbf{1}$

Table 2: Table of values of  $\Lambda_{d,d}(s * Q^{p-1})$

State $s$	$\Lambda_{d,d}(s * Q^{p-1})$
$xy$	$a_{p-1} xy$ for $\mathbf{d} = \mathbf{0}$ $1$ for $\mathbf{d} = \mathbf{1}$ $\sum_{k \geq 0} \binom{p-1-k}{d-1-2k} \binom{d-1}{k} (-1)^{d+1}$ for $\mathbf{2} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{3}$ $\frac{1}{2}(c_{p+1} - b_{p+1})$ for $\mathbf{d} = \mathbf{p} - \mathbf{2}$ $b_p$ for $\mathbf{d} = \mathbf{p} - \mathbf{1}$
$x^2y^3$	$-\sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k+1} xy + x^2y^2 + x^2y^3$ $= (-a_p - b_p + 1) xy + x^2y^2 + x^2y^3$ for $\mathbf{d} = \mathbf{0}$ $\sum_{k \geq 0} \binom{p-1-k}{k} \binom{p-1}{k+1} xy = -a_{p-1} xy$ for $\mathbf{d} = \mathbf{1}$ $0$ for $\mathbf{d} = \mathbf{2}$ $\sum_{k \geq 0} \binom{p-1-k}{d-2-2k} \binom{d-2}{k+1} (-1)^d$ for $\mathbf{3} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{2}$ $-\frac{1}{2}(c_{p+1} + b_{p+1})$ for $\mathbf{d} = \mathbf{p} - \mathbf{1}$
$x^2y^4$	$-\sum_{k \geq 0} \binom{p-1-k}{k+1} \binom{p-2}{k+2} xy - 2x^2y^2 - 2x^2y^3$ $= (2a_p + b_p - 2)xy - 2x^2y^2 - 2x^2y^3$ for $\mathbf{d} = \mathbf{0}$ $\sum_{k \geq 0} \binom{p-1-k}{k} \binom{p-1}{k+2} xy = a_{p-1} xy$ for $\mathbf{d} = \mathbf{1}$ $0$ for $\mathbf{2} \leq \mathbf{d} \leq \mathbf{3}$ $\sum_{k \geq 0} \binom{p-1-k}{d-2-2k} \binom{d-2}{k+2} (-1)^d$ for $\mathbf{4} \leq \mathbf{d} \leq \mathbf{p} - \mathbf{3}$ $-\sum_{k \geq 0} \binom{p-1-k}{k+3} \binom{p-4}{k+2} + xy + xy^2$ for $\mathbf{d} = \mathbf{p} - \mathbf{2}$ $\sum_{k \geq 0} \binom{p-1-k}{k+2} \binom{p-3}{k+2} - xy - xy^2$ $= \frac{1}{2}(c_{p+1} + 3b_{p+1} + 2a_{p+1} - 2) - xy - xy^2$ for $\mathbf{d} = \mathbf{p} - \mathbf{1}$

Table 3: Table of values of  $\Lambda_{d,d}(s * Q^{p-1})$

$$= \sum_{k, l \geq 0} \binom{p-1}{k} \binom{p-1-k}{l} x^{2k+l} y^k (y+1)^{2k+l} x^l (-1)^{k+l}.$$

Using the identity

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

we then have

$$Q^{p-1}(x, y) = \sum_{k, l, m \geq 0} \binom{p-1-k}{l} \binom{2k+l}{m} (-1)^l x^{2k+l} y^{k+m}. \quad (17)$$

We define  $a_{k, l, m}$  by

$$a_{k, l, m} := \binom{p-1-k}{l} \binom{2k+l}{m} (-1)^l. \quad (18)$$

We then have

$$Q^{p-1}(x, y) = \sum_{k, l, m \geq 0} a_{k, l, m} x^{2k+l} y^{k+m} = \sum_{i, j \geq 0} b_{i, j} x^i y^j$$

where

$$b_{i, j} = \sum a_{k, l, m}$$

and the sum is over  $k, l, m \geq 0 : 2k+l = i, k+m = j$ . So,

$$b_{i, j} = \sum_{k \geq 0} \binom{p-1-k}{i-2k} \binom{i}{j-k} (-1)^i. \quad (19)$$

We will next calculate the effect of the Cartier operator  $\Lambda_{d, d}$  on a general monomial  $x^r y^t$ . For  $0 \leq d \leq p-1$  we have

$$\Lambda_{d, d}(x^r y^t Q^{p-1}) = \Lambda_{d, d}\left(\sum_{i, j \geq 0} b_{i, j} x^{i+r} y^{j+t}\right).$$

$$= \Lambda_{d, d}\left(\sum_{i \geq r, j \geq t} c_{i, j} x^i y^j\right) = \sum_{i \geq r, j \geq t} c_{pi+d, pj+d} x^i y^j$$

where  $c_{i, j}$  is defined by  $c_{i, j} := b_{i-r, j-t}$ . So

$$\Lambda_{d, d}(x^r y^t Q^{p-1}) = \sum_{i \geq r, j \geq t} \left( \sum_{k \geq 0} \binom{p-1-k}{pi+d-r-2k} \binom{pi+d-r}{pj+d-t-k} (-1)^{i+d+r} \right) x^i y^j. \quad (20)$$

The sum above is finite as the indices  $i$  and  $j$  satisfy the restrictions

$$r \leq pi + d \leq 2(p-1) + r \quad \text{and} \quad t \leq pj + d \leq 3(p-1) + t.$$

We will first look at the monomial 1, for which we have  $r = t = 0$ . For this choice of  $r$  and  $t$  we need to determine

$$\binom{p-1-k}{pi+d-2k} \binom{pi+d}{pj+d-k} (-1)^{i+d}.$$

for all possible choices of  $i$  and  $j$ . In this case it turns out all terms are 0 mod  $p$  except for the  $i = j = 0$  term. So, for  $0 \leq d \leq p-1$ ,

$$\Lambda_{d,d}(1 * Q^{p-1}) = \sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k} (-1)^d.$$

For  $d = p-2$  this sum reduces to  $b_p$  as

$$\begin{aligned} \sum_{k \geq 0} \binom{p-1-k}{p-2-2k} \binom{p-2}{k} (-1)^{p-2} &= \sum_{k \geq 0} \binom{p-1-k}{k+1} (-1)^{k+1} (k+1) \\ &= \sum_{k \geq 1} \binom{p-k}{k} (-1)^k k \\ &= b_p. \end{aligned}$$

For  $d = p-1$  the sum is  $a_{p-1}$  as

$$\sum_{k \geq 0} \binom{p-1-k}{p-1-2k} \binom{p-1}{k} (-1)^{p-1} = \sum_{k \geq 0} \binom{p-1-k}{k} (-1)^k = a_{p-1}$$

The remaining cases can be treated similarly giving the results stated in table 2 and table 3.

## 6 Constructing the automata for $M_n \mod p$

In this section we will describe the states and transitions of the automata for  $M_n \mod p$ . These are summarised in table 4 for the case  $p \equiv 1 \mod 6$  and tables 5

$d$	$s_1$	$s_2$ $2x^2y^2(y+1) + xy$	1	$-xy(y+1)$	$xy(y+1) + 2$
$d = 0$	$s_2$	$s_2$	1	0	2
$d = 1$	1	1	1	$c$	$c$
$2 \leq d \leq p-3$	$c$	$c$	$c$	$c$	$c$
$d = p-2$	$-xy(y+1)$	$c$	$c$	$c$	0
$d = p-1$	$xy(y+1) + 2$	$c$	1	$-xy(y+1)$	$xy(y+1) + 2$

Table 4: Table of states and transitions for  $p \equiv 1 \pmod{6}$ .

and 6 for the case  $p \equiv -1 \pmod{p}$ . A 'c' appearing in the tables represents a constant state (constant polynomial). The value of  $c$  depends on  $d$  and  $p$ . For given  $d : 0 \leq d \leq p-1$  and state  $s$  the tables give the state equal to

$$\Lambda_{d,d}(s * Q^{p-1}).$$

The transition  $(s, d) \rightarrow \Lambda_{d,d}(s * Q^{p-1})$  is then part of the automaton.

The behaviour of the automaton depends on the value of  $p \pmod{6}$ . When  $p \equiv 1 \pmod{6}$  there are up to  $p+4$  states consisting of the  $p$  constant polynomials modulo  $p$  and the 4 polynomials

$$s_1 = -2x^2y^3(y+1) - xy^2 + y, \quad s_2 = x^2y^2(y+1) + xy, \quad -xy(y+1), \quad x^2 + xy^2 + 2.$$

When  $p \equiv -1 \pmod{6}$  there are up to  $p+6$  states consisting of the  $p$  constant polynomials modulo  $p$  and the 6 polynomials

$$s_1, \quad s_2, \quad -xy(y+1) - 1, \quad xy(y+1) - 1, \quad xy(y+1), \quad xy(y+1) + 2.$$

It is unclear whether all  $p$  constant polynomials always appear as states.

$d$	$s_1$	$s_2$ $2x^2y^2(y+1) + xy$	1	$-xy(y+1) - 1$
$d = 0$	$s_2$	$s_2$	1	$-1$
$d = 1$	1	1	1	$c$
$2 \leq d \leq p-4$	$c$	$c$	$c$	$c$
$d = p-3$	$c$	$c$	$c$	0
$d = p-2$	$-xy(y+1) - 1$	$c$	$c$	$c$
$d = p-1$	$xy(y+1) - 1$	$c$	$-1$	$-xy(y+1)$

Table 5: Table of states and transitions for  $p \equiv -1 \pmod{6}$ .

$d$	$xy(y+1) - 1$	$-xy(y+1)$	$xy(y+1) + 2$
$d = 0$	$-1$	0	2
$1 \leq d \leq p-3$	$c$	$c$	$c$
$d = p-2$	$c$	$c$	0
$d = p-1$	$xy(y+1) + 2$	$-xy(y+1) - 1$	$xy(y+1) - 1$

Table 6: Table of states and transitions for  $p \equiv -1 \pmod{6}$ .

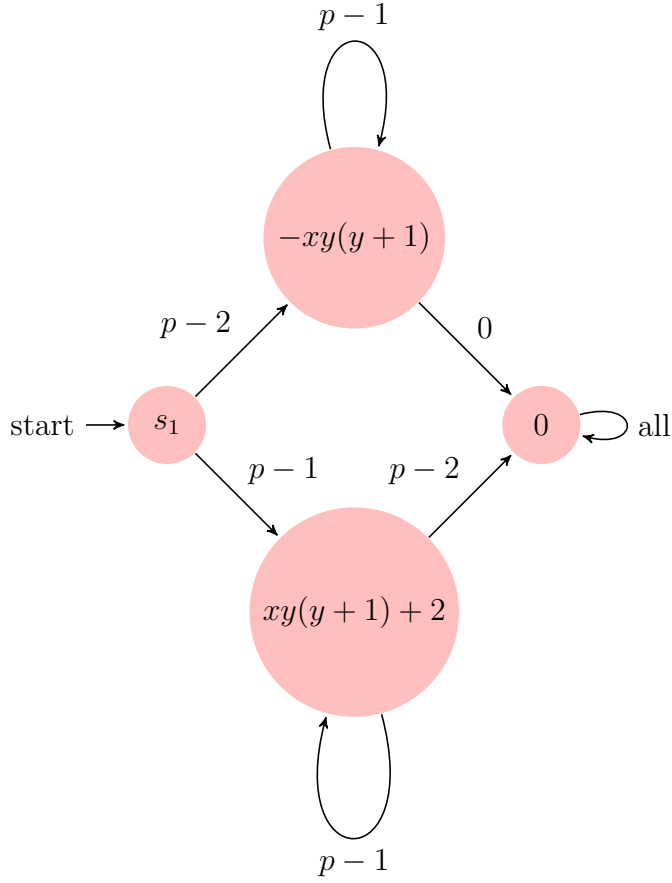


Figure 2: Partial state diagram for  $M_n \bmod p$  when  $p \equiv 1 \bmod p$ .

Figures 2, 3 and 4 provide an alternative pictorial summary of the automata for  $M_n \bmod p$ .

The calculation of the states will rely on the data contained in table 2 and table 3. As mentioned earlier, the initial state  $s_1$  for the automata is the polynomial defined in equation (14). The second state  $s_2$  is then given by

$$\begin{aligned}
 s_2 &= \Lambda_{0,0}(s_1 * Q(x, y)^{p-1}) \\
 &= \Lambda_{0,0}(y(1 - xy - 2x^2y^2 - 2x^2y^3) * Q(x, y)^{p-1}) \\
 &= a_{p-1}xy - 2 \left( (-a_p - b_p + 1)xy + x^2y^2 + x^2y^3 \right) - 2 \left( (2a_p + b_p - 2)xy - 2x^2y^2 - 2x^2y^3 \right)
 \end{aligned}$$

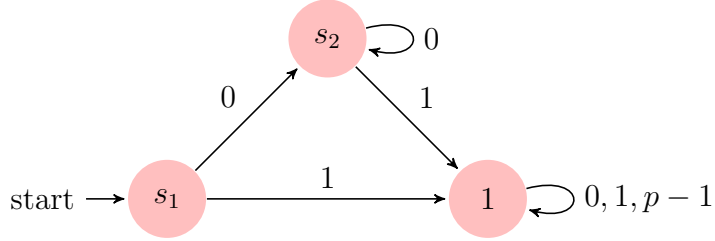


Figure 3: Another part of the state diagram for  $M_n \bmod p$  when  $p \equiv 1 \pmod{p}$ .

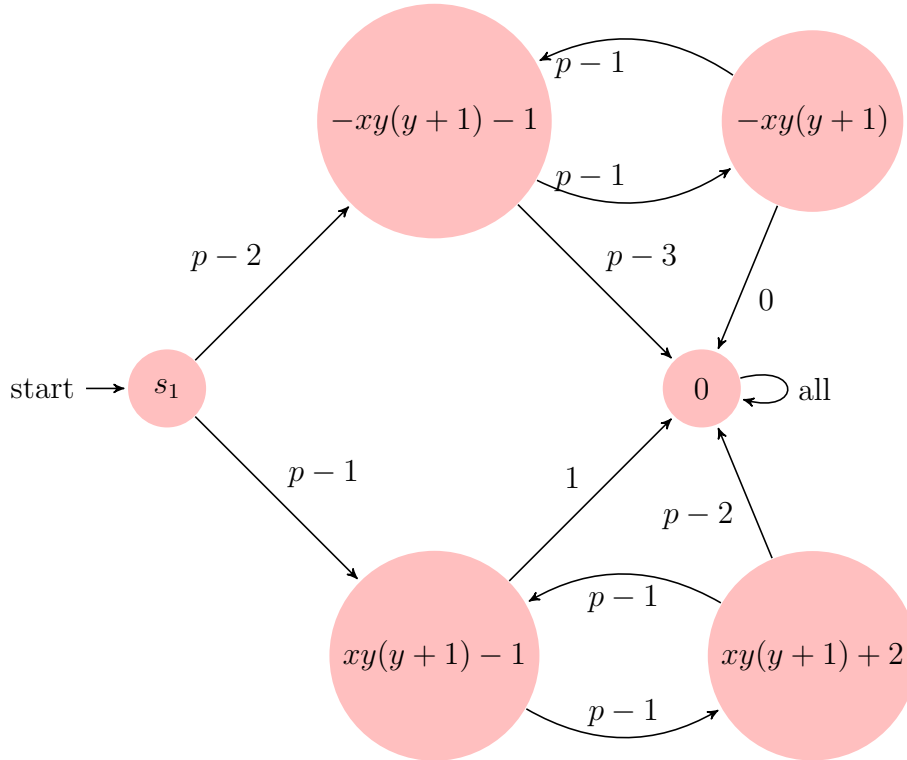


Figure 4: Partial state diagram for  $M_n \bmod p$  when  $p \equiv -1 \pmod{p}$ .



$$\begin{aligned}
&= 2x^2y^2 + 2x^2y^3 + (a_{p-1} - 2a_p + 2)xy \\
&= 2x^2y^2 + 2x^2y^3 + xy
\end{aligned}$$

from equation (9).

The next interesting state is  $\Lambda_{p-2,p-2}(s_1 * Q(x, y)^{p-1})$ . We have

$$\begin{aligned}
\Lambda_{p-2,p-2}(s_1 * Q(x, y)^{p-1}) &= -a_p - b_p + 1 + xy + xy^2 + \frac{1}{2}(b_{p+1} + c_{p+1}) \\
&+ 2\left(\sum_{k \geq 0} \binom{p-1-k}{k+3} \binom{p-4}{k+1}\right) + 2\left(\sum_{k \geq 0} \binom{p-1-k}{k+3} \binom{p-4}{k+2} - xy - xy^2\right) \\
&= -xy(y+1) - a_p - b_p + 1 + \frac{1}{2}(b_{p+1} + c_{p+1}) + 2\left(\sum_{k \geq 0} \binom{p-1-k}{k+3} \binom{p-3}{k+2}\right).
\end{aligned}$$

Now since

$$\begin{aligned}
\sum_{k \geq 0} \binom{p-1-k}{k+3} \binom{p-3}{k+2} &= \frac{1}{2} \sum_{k \geq 0} \binom{p-1-k}{k+3} (-1)^k (k+4)(k+3) \\
&= -\frac{1}{2} \sum_{k \geq 3} \binom{p+2-k}{k} (-1)^k k(k+1) \\
&= -\frac{1}{2}(c_{p+2} + (p+1) - 4 \binom{p}{2}) + b_{p+2} + (p+1) - 2 \binom{p}{2} \\
&\equiv -\frac{1}{2}(c_{p+2} + b_{p+2} + 2) \pmod{p}
\end{aligned}$$

we have

$$\Lambda_{p-2,p-2}(s_1 * Q(x, y)^{p-1}) = -xy(y+1) - a_p - b_p - 1 + \frac{1}{2}b_{p+1} + \frac{1}{2}c_{p+1} - b_{p+2} - c_{p+2}.$$

So, using equation 11 and operating modulo  $p$ ,

$$\Lambda_{p-2,p-2}(s_1 * Q(x, y)^{p-1}) = \begin{cases} -xy(y+1) & \text{if } p \equiv 1 \pmod{6}; \\ -xy(y+1) - 1 & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

The next state to appear is  $\Lambda_{p-1,p-1}(s_1 * Q(x, y)^{p-1})$ .

$$\Lambda_{p-1,p-1}(s_1 * Q(x, y)^{p-1}) = -a_{p-1} - (-a_p - b_p + 1 + xy + xy^2) + (c_{p+1} + b_{p+1})$$

$$\begin{aligned}
& -(c_{p+1} + 3b_{p+1} + 2a_{p+1} - 2) + 2(xy + xy^2) \\
= & xy(y+1) - a_{p-1} + a_p + b_p - 2b_{p+1} - 2a_{p+1} + 1 \\
= & \begin{cases} xy(y+1) + 2 & \text{if } p \equiv 1 \pmod{6}; \\ xy(y+1) - 1 & \text{if } p \equiv -1 \pmod{6}; \end{cases}
\end{aligned}$$

using equation (12). The transitions  $\Lambda_{d,d}(s_2 * Q(x, y)^{p-1})$  all produce constant states except for  $\Lambda_{0,0}(s_2 * Q(x, y)^{p-1})$ .

$$\begin{aligned}
\Lambda_{0,0}(s_2 * Q(x, y)^{p-1}) &= a_{p-1}xy + 2b_pxy + 2(-a_p - b_p + 1)xy + 2x^2y^2 + 2x^2y^3 \\
&= (a_{p-1} - 2a_p + 2)xy + 2x^2y^2 + 2x^2y^3 \\
&= s_2
\end{aligned}$$

using equation (9). In order to complete the entries from tables 4, 5 and 6 it is enough to examine the transitions  $\Lambda_{d,d}(xy(y+1) * Q(x, y)^{p-1})$  (noting that  $xy(y+1)$  is not actually a state). A similar calculation to the one for  $\Lambda_{p-2,p-2}(s_1 * Q(x, y)^{p-1})$  can be used to show that

$$\begin{aligned}
\Lambda_{p-3,p-3}(xy(y+1) * Q(x, y)^{p-1}) &= \frac{1}{2}(b_{p+2} - c_{p+2}); \\
\Lambda_{p-3,p-3}(1 * Q(x, y)^{p-1}) &= \frac{1}{2}(c_{p+1} - b_{p+1}).
\end{aligned}$$

We then have

$$\begin{aligned}
\Lambda_{p-3,p-3}((-xy(y+1) - 1) * Q(x, y)^{p-1}) &= \frac{1}{2}((c_{p+2} - c_{p+1}) - (b_{p+2} - b_{p+1})) \\
&= \frac{1}{2}((-a_p - 2b_p - c_p) - (-a_p - b_p)) \\
&= -\frac{1}{2}(c_p + b_p) \\
&\equiv 0 \pmod{p}
\end{aligned}$$

using equation (10).

We also have

$$\Lambda_{p-1,p-1}(xy(y+1) * Q(x, y)^{p-1})$$

$$\begin{aligned}
&= b_p + (-a_p - b_p + 1) + xy + xy^2 \\
&= 1 - a_p + xy(y + 1) \\
&= \begin{cases} xy(y + 1) & \text{if } p \equiv 1 \pmod{6}; \\ xy(y + 1) + 1 & \text{if } p \equiv -1 \pmod{6}. \end{cases}
\end{aligned}$$

## 7 Conclusions

The values of  $n$  mentioned in table 1 for which  $M_n \equiv 0 \pmod{p}$  can be immediately derived from an inspection of tables 4, 5 and 6 and the associated state diagrams in figures 2, 3 and 4. A lower bound for the asymptotic density of the set  $S_p(0)$  can then

be derived from the following result from [1]

**Theorem 5.** *Let*

$$S(q, r, s, t) = \{(qi + r)q^{sj+t} : i, j \in \mathbb{N}\}$$

and

$$S'(q, r, s, t) = \{(qi + r)q^{sj+t} : i, j \in \mathbb{N}, j \geq 1\}$$

for integers  $q, r, s, t \in \mathbb{Z}$  with  $q, s > 0$ ,  $t \geq 0$  and  $0 \leq r < q$ . Then the asymptotic density of the set  $S$  is  $(q^{t+1-s}(q^s - 1))^{-1}$ . The asymptotic density of the set  $S'$  is  $(q^{t+1}(q^s - 1))^{-1}$ .

From above we know that if  $p \equiv 1 \pmod{p}$  then  $M_n \equiv 0 \pmod{p}$  when  $n$  is in the forms

$$\begin{aligned}
n &= (pi + 1)p^k - 2 \text{ for } i \geq 0 \text{ and } k \geq 1. \\
n &= (pi + p - 1)p^k - 1 \text{ for } i \geq 0 \text{ and } k \geq 1.
\end{aligned}$$

Each of the 2 forms has asymptotic density  $\frac{1}{p(p-1)}$ . Therefore, when  $p \equiv 1 \pmod{6}$  the asymptotic density of  $S_p(0) \geq \frac{2}{p(p-1)}$ .

For  $p \equiv -1 \pmod{6}$  there are 4 forms of numbers to consider. These are

$$\begin{aligned}
n &= (pi + 1)p^{2k} - 2 \text{ for } i \geq 0 \text{ and } k \geq 1. \\
n &= (pi + p - 2)p^{2k+1} - 2 \text{ for } i \geq 0 \text{ and } k \geq 0. \\
n &= (pi + 2)p^{2k+1} - 1 \text{ for } i \geq 0 \text{ and } k \geq 0. \\
n &= (pi + p - 1)p^{2k} - 1 \text{ for } i \geq 0 \text{ and } k \geq 1.
\end{aligned}$$

The first and fourth forms have asymptotic density  $(p(p^2 - 1))^{-1}$ . The second and third forms have asymptotic density  $(p^2 - 1)^{-1}$ . Therefore, when  $p \equiv -1 \pmod{6}$  the asymptotic density of  $S_p(0)$  is again  $\geq \frac{2}{p(p-1)}$ .

Tables 4, 5 and 6 and the state diagrams in Figures 2, 3 and 4 can be used to determine which numbers  $n$  have  $M_n \equiv x \pmod{p}$  for other values of  $x$ . For example, figure 3 shows that if  $p \equiv 1 \pmod{6}$  then  $M_n \equiv 1 \pmod{p}$  when the base  $p$  representation of  $n$  contains only 0's and 1's. Figure 2 shows that if  $p \equiv 1 \pmod{6}$  then  $M_n \equiv 2 \pmod{p}$  when  $n = p^k - 1$  for some  $k \in \mathbb{N}$ .

As mentioned in the introduction results on forbidden residues of  $M_n \pmod{p^k}$  have been proved for some primes  $p$  and some  $k \geq 2$ . In order to whether there are forbidden residues  $\pmod{p}$  itself the constant states of the automata would need to be examined. Since

$$\Lambda_{d,d}(1 * Q(x, y)^{p-1}) = c(p, d) := \sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k} (-1)^d$$

in order to show there are no forbidden residues  $\pmod{p}$  it is sufficient to show that the set

$$\{c(p, d) : 0 \leq d \leq p-1\}$$

generates  $(\frac{\mathbb{Z}}{p\mathbb{Z}})^\times$ .

As shown in [2] the asymptotic density of  $S_p(0)$  is actually 1 for some primes, e.g.  $p = 7, 17, 19$ . In these cases, there is a  $d : 2 \leq d \leq p-2$  such that

$$\Lambda_{d,d}(1 * Q(x, y)^{p-1}) = 0.$$

It then follows that

$$\Lambda_{d,d}(c * Q(x, y)^{p-1}) = 0$$

for all constants  $c$ . As a result, any  $n$  which has a base- $p$  representation containing 2 or more digits  $d$  satisfies  $M_n \equiv 0 \pmod{p}$ . The asymptotic density of this set is 1. It would therefore be of interest to determine for which  $p$  and  $d$

$$\sum_{k \geq 0} \binom{p-1-k}{d-2k} \binom{d}{k} \equiv 0 \pmod{p}.$$

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